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An Orientation Theorem for Graphs

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We characterize the class of graphs in which the edges can be oriented in such a way that going along any circuit in the graph, the number of forward edges minus the number of backward edges is equal to 0, 1, or −1. The result follows by applying Tutte's characterization of regular matroids to a certain binary matroid associated to a graph.

1. AN ORIENTATION THEOREM

A directed graph $D = (V(D), A(D))$ has *discrepancy* k if for each circuit C in D the number of forward arcs and the number of backward arcs differ by at most k. An *orientation* of an undirected graph $G = (V(G), E(G))$ is a directed graph D obtained from G by replacing each edge in G by a directed edge (arc). Obviously a graph G has an orientation of discrepancy 0 if and only if G is bipartite. In this paper we extend this fact to the following.

ORIENTATION THEOREM. *Let G be an undirected graph. Then the following are equivalent*:

(i) *G has an orientation of discrepancy* 1;

(ii) *G contains neither an odd-K_4 nor an odd-K_3^2 as a subgraph.*

Here an *odd-K_4* and an *odd-K_3^2* are undirected graphs as depicted in Fig. 1.

Clearly, an orientation of discrepancy 1 is never unique: reversing the orientation of all the arcs in a directed cut—called *directed-cut switching*—preserves the discrepancy. Also *total switching*, reversing the orientation of all the arcs, does not change the discrepancy. The same is true if we apply *block switching*, that is, the a total switching of a single block of G. Recall that a *block* (or *2-connected component*) of G is an equivalence class of the

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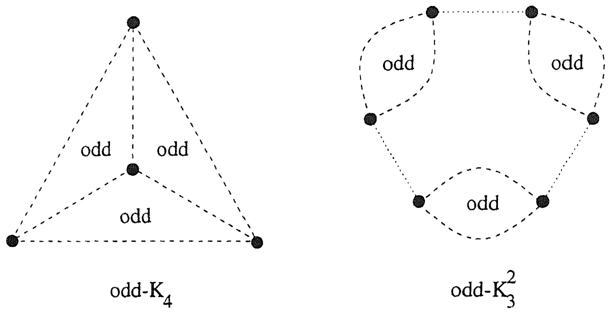

FIG. 1. Dashed and dotted lines denote pairwise openly disjoint paths. Dashed lines correspond to paths with at least one edge, whereas dotted lines may have length 0. The word **odd** in a face indicates that the length of the bounding circuit is odd.

equivalence relation on $E(G)$ in which two edges are related if there exists a circuit containing both. Two orientations are *switching equivalent* if one can be obtained from the other by a series of block switchings and directed-cut switchings.

UNIQUENESS THEOREM. *All orientations of discrepancy* 1 *in an undirected graph G are switching equivalent.*

The proof of the Orientation Theorem is in Section 2, and of the Uniqueness Theorem in Section 3. In Section 4 we consider the problem of actually finding an orientation of discrepancy 1 and the problem of checking whether a given orientation has discrepancy 1. In Section 5 we discuss some applications of the Orientation Theorem. Finally, in Section 6, we state a dual version of the Orientation Theorem.

2. PROOF OF THE ORIENTATION THEOREM

We prove the Orientation Theorem by applying Tutte's forbidden minor characterization of regular matroids [15] to a certain binary matroid, $\mathscr{S}(G)$, associated with a graph G. Therefore we present in Section 2.1 a short introduction to notions relevant for Tutte's Theorem. As the class of binary matroids $\mathscr{S}(G)$ is not closed under taking minors, we extend in Section 2.2 our domain from "graphs" to "signed graphs."

2.1. *Binary and Regular Matroids*

A matroid is *binary* if it is representable by the columns of a *binary matrix*, i.e., a matrix with entries in $GF(2)$. A matroid is *regular* if it is representable over the reals by a unimodular matrix with full row rank. (An $m \times n$ matrix is *unimodular* if all its enties are integral and all its $m \times m$ subdeterminants are 0, 1, or -1. A matrix is *totally unimodular* if all its subdeterminants are 0, 1, or -1.) Regular matroids are binary. (The reduction modulo 2 of a unimodular representation of full row rank, yields a binary representation of the same matroid.) But not all binary matroids are regular. In particular the following two binary matrices represent nonregular matroids (F_7, F_7^*, respectively):

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} (F_7); \qquad \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} (F_7^*).$$

The gist of Tutte's Theorem is that in a sense the two examples above are the only nonregular binary matroids. To express precisely what we mean by that, we recall the notion of minors. (We restrict ourselves to representable matroids.)

Let $\mathcal{M}$ be a matroid represented over some field $\mathcal{F}$ by matrix M. If $e \in \mathcal{M}$ then m_e denotes the column of M corresponding to e.

Deleting e. Let M' be obtained from M by deleting m_e. The matroid $\mathcal{M} \backslash e$, obtained from $\mathcal{M}$ by *deleting* e, is the matroid represented by M'. (Note that $\mathcal{M} \backslash e$, depends only on $\mathcal{M}$ and e, not on the actual representation M.)

Contracting e. Let M' be obtained from M by *row operations* (= adding rows to other rows and *scaling* rows, i.e., multiplying them with scalars) such that m'_e is a unit column or $m'_e = 0$. Let M'' be obtained from M' by deleting m'_e and the row of M' containing a 1 in m'_e. The matroid $\mathcal{M}/e$, obtained from $\mathcal{M}$ by *contracting* e, is the matroid represented by M''. (Note that $\mathcal{M}/e$ depends only on $\mathcal{M}$ and e, not on M, nor on M'. Moreover note that if $m_e = 0$ then $\mathcal{M} \backslash e = \mathcal{M}/e$.)

Minors. A *minor* of $\mathcal{M}$ is a matroid obtained from $\mathcal{M}$ by a series of deletions and contractions.

If M is a unimodular matrix of full row rank, then for the row operations as meant in defining contraction, we can use *unimodular row operations*, where scaling is restricted to multiplication by -1. Unimodular row operations preserve unimodularity. Hence the class of regular matroids is closed under taking minors. In fact, we have:

TUTTE'S THEOREM [15]. *Let $\mathcal{M}$ be a binary matroid. Then $\mathcal{M}$ is regular if and only if $\mathcal{M}$ has neither F_7 nor F_7^* as minors.*

A short proof of this theorem can be found in Gerards [5]. For the theory of regular matroids we refer to Bixby [1], Oxley [11], Schrijver [12], Truemper [14], Tutte [15–17], and Welsh [18].

We have already mentioned that unimodular representations can be turned easily into binary ones. If a regular matroid is given by a binary representation M, it may be—and in our case it is—useful to have a representation over the reals (not necessarily unimodular) in which M is still easily recognizable.

LEMMA 1. *Let M be a binary matrix. If M represents a regular matroid $\mathcal{M}$, then there exists a $\{0, \pm 1\}$ matrix $\tilde{M} \equiv M \pmod 2$ which represents $\mathcal{M}$ over the reals. Moreover for each with $Mx = 0$, there exists a $\{0, \pm 1\}$ vector $\tilde{x} \equiv x \pmod 2$ with $\tilde{M}\tilde{x} = 0$.*

Proof. Assume M is an $m \times n$ matrix. Let M_1 be an $m \times r$ submatrix of M with linear independent columns, where r denotes the rank of M over $GF(2)$. We may assume

(1) $M = [M_1 \mid M_2]$.

Then $\mathcal{M}$ is also represented by

(2) $[I \mid C]$ where C is uniquely determined by $M_2 = M_1 C$.

Let B be a unimodular representation of $\mathcal{M}$, such that B has full row rank. Assume $B = [B_1 \mid B_2]$, where the columns of B_1 corresponds to the columns of I in $[I \mid C]$. Then B_1 is nonsingular and M is representable over the reals by

(3) $[I \mid B_1^{-1} B_2] =: [I \mid D]$.

It is easy to see that $[I \mid D]$ is unimodular too. Hence D is totally unimodular. Moreover, as (2) and (3) represent the same matroid,

(4) $D \equiv C \pmod 2$.

Now we use a well-known fact about totally unimodular matrices, due to Ghouila-Houri [8]. Let k denote the number of columns of D, i.e., $k = n - r$.

(5) For each vector $x \in GF(2)^r$ there exists an $\tilde{x} \in \{0, \pm 1\}^r$, with $\tilde{x} \equiv x \pmod 2$ and $\tilde{x}^T D \in \{0, \pm 1\}^k$.

(In fact (5) is equivalent with D being totally unimodular, but we only use its necessity. Proving (5) is quite easy; cf. Schrijver [12, p. 269].)

Applying (5) to the rows of M_1, yields a $\{0, \pm 1\}$ matrix $\tilde{M}_1 \equiv M_1$ such that $\tilde{M}_1 D$ is a $\{0, \pm 1\}$ matrix. The columns of $\tilde{M}_1$ are linearly independent. [Indeed, let M_{11} be an $r \times r$ nonsingular submatrix of M_1, and let $\tilde{M}_{11}$ be the corresponding submatrix of $\tilde{M}_1$. Then $\det \tilde{M}_{11} \equiv \det M_{11} \neq 0$.] So we get that

(6) $\tilde{M} := \tilde{M}_1 [I \mid D]$ is a $\{0, \pm 1\}$ matrix representing $\mathcal{M}$ over the reals,

and that

$$(7) \quad \tilde{M} = \tilde{M}_1[I \mid D] \equiv M_1[I \mid C] = M \pmod 2.$$

Next we prove the second part of the lemma. Let x be such that $Mx = 0$. Hence $x_1 + Cx_2 = 0$ with $x^\top = [x_1^\top, x_2^\top]$. Applying (5) to x_2 instead of x and $D^\top$ instead of D yields a $\{0, \pm 1\}$ vector $\tilde{x}_2 \equiv x_2$ such that $\tilde{x}_1 := -D\tilde{x}_2$ is a $\{0, \pm 1\}$ vector, too. Hence $\tilde{x} := [\tilde{x}_1^\top, \tilde{x}_2^\top]^\top$ is a $\{0, \pm 1\}$ vector with $\tilde{x} = [\tilde{x}_1^\top, \tilde{x}_2^\top]^\top = [-\tilde{x}_2^\top D^\top, \tilde{x}_2^\top]^\top \equiv [-x_2^\top C^\top, x_2^\top]^\top = [x_1^\top, x_2^\top]^\top = x \pmod 2$ and $\tilde{M}\tilde{x} = \tilde{M}_1(\tilde{x}_1 + D\tilde{x}_2) = 0$. ∎

Remark. In fact, existence of $\tilde{M}$ as in Lemma 1 is also sufficient for regularity of $\mathcal{M}$.

2.2. *Signed Graphs*

A *signed graph* is a pair (G, Σ), where G is an undirected graph and Σ a subset of $E(G)$. Edges in Σ are called *odd*; the other edges *even*. A circuit in G is *odd* (*even*) if it contains an odd (even) number of odd edges. We extend the notions odd-K_4 and odd-K_3^2 to the setting of signed graphs. A signed graph is an *odd-K_4* or an *odd-K_3^2* if it is of the form as depicted in Fig. 1. Now the word **odd** in a face means that that face is bounded by a circuit which is odd in the signed graph. Associated to a signed graph we consider the binary matroid $\mathcal{S}(G, \Sigma)$ which is represented over $GF(2)$ by the columns of the matrix

$$S := \left[\begin{array}{c|c} 1 & \chi_\Sigma^\top \\ \hline 0 & \\ \vdots & M_G \\ 0 & \end{array}\right],$$

where M_G denotes the node-edge incidence matrix of G. (Rows of M_G correspond to nodes of G; columns to edges.) χ_Σ denotes the characteristic row vector of Σ as a subset of $E(G)$. The special element of $\mathcal{S}(G, \Sigma)$ not in $E(G)$, corresponding to the first column of S, will be denoted by σ. Note that $\mathcal{S}(G, \Sigma)$ does not depend so much on Σ but rather on which circuits are odd and which circuits are even with respect to Σ. This means that $\mathcal{S}(G, \Sigma)$ is invariant under *re-signing on* $U \subseteq V(G)$, i.e., replacing Σ by the symmetric difference $\Sigma \,\Delta\, \delta(U)$ of Σ ($\delta(U) := \{uv \in E(G) \mid u \in U; v \notin U\}$). We want to apply Tutte's Theorem to the matroid $\mathcal{S}(G, \Sigma)$. Hence we must know its minors.

Deletion of an edge. Deleting $e \in E(G)$ from $\mathcal{S}(G, \Sigma)$ amounts to just deleting edge e from the graph G.

Contraction of an even edge. Contracting $e \in E(G) \backslash \Sigma$ in $\mathcal{S}(G, \Sigma)$ amounts to just contracting edge e in the graph G.

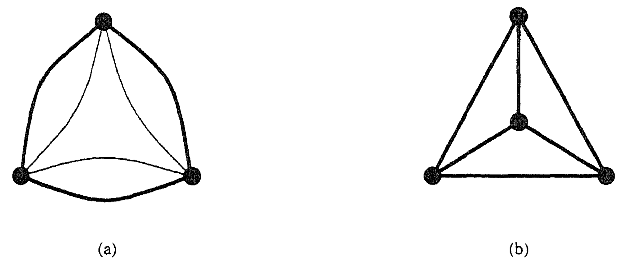

FIG. 2. Bold edges are in Σ.

Contraction of an odd edge (not being a loop). If $e = uv$ is an odd edge (with $u \neq v$), then contracting e in $\mathscr{S}(G, \Sigma)$ corresponds to re-signing (G, Σ) on $\{u\}$, and then contracting the (now even) edge e in the graph G.

Contraction of σ and of odd loops. $\mathscr{S}(G, \Sigma)/\sigma = \mathscr{M}(G)$, the *cycle matroid* of G. The same matroid results (upto isomorphism) when we contract an odd loop in $\mathscr{S}(G, \Sigma)$.

Deletion of σ. $\mathscr{S}(G, \Sigma)\backslash\sigma$ is the *even cycle matroid* $\mathscr{E}(G, \Sigma)$ of (G, Σ).

LEMMA 2. *Let (G, Σ) be a signed graph. Then (G, Σ) contains no odd-K_4 and no odd-K_3^2 if and only if $\mathscr{S}(G, \Sigma)$ has neither F_7 nor F_7^* as a minor.*

Proof. It is easy to prove that (G, Σ) contains no odd-K_4 and no odd-K_3^2 if and only if it cannot be reduced to any of the two signed graphs in Fig. 2 by deleting edges, contracting even edges, and re-signing. Moreover, we have

— $\mathscr{S}(G, \Sigma)$ is isomorphic to F_7 if and only if (G, Σ) is the signed graph in Fig. 2a.

— $\mathscr{S}(G, \Sigma)$ is isomorphic to F_7^* if and only if (G, Σ) is (upto re-signing) the signed graph in Fig. 2b.

— $\mathscr{E}(G, \Sigma)$ is isomorphic to F_7 if and only if (G, Σ) is one of the two signed graphs (a) and (b) in Fig. 3.

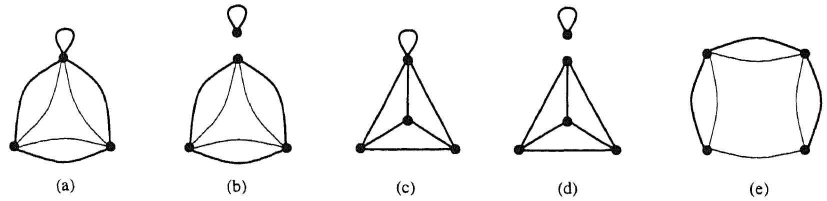

FIG. 3. Bold edges are in Σ.

— $\mathscr{E}(G, \Sigma)$ is isomorphic to F_7^* if and only if (G, Σ) is (upto re-signing) one of the three signed graphs (c), (d), and (e) in Fig. 3.

From the above, the lemma easily follows. ∎

2.3. *Proof*

Observe that neither an odd-K_4 nor an odd-K_3^2 have an orientation of discrepancy 1. Hence to prove the Orientation Theorem we may assume that G contains no odd-K_4 and no odd-K_3^2. Let $\mathscr{S}(G) := \mathscr{S}(G, E(G))$, i.e., $\mathscr{S}(G)$ is represented over $GF(2)$ by

$$S := \left[\begin{array}{c|ccc} 1 & 1 & 1 & \cdots & 1 \\ \hline 0 & & & & \\ \vdots & & M_G & & \\ 0 & & & & \end{array}\right].$$

By Lemma 2 and Tutte's Theorem $\mathscr{S}(G)$ is regular. Hence, by Lemma 1 it can be represented over the reals by a matrix

$$\tilde{S} = \left[\begin{array}{c|ccc} 1 & 1 & 1 & \cdots & 1 \\ \hline 0 & & & & \\ \vdots & & N_G & & \\ 0 & & & & \end{array}\right],$$

where N_G is a $\{0, \pm 1\}$ matrix with $N_G \equiv M_G \pmod 2$. That the top row of $\tilde{S}$ consists of ones only can easily be achieved by multiplying some of the columns of $\tilde{S}$ by -1.

CLAIM. *We may assume that each (nonzero) column of N_G has exactly one* 1 *and one* -1.

Proof. Let $F \subseteq E(G)$ be a forest of maximum cardinality. Let N_F be the submatrix of N_G consisting of the columns corresponding with the edges in F. As F is a forest, we can multiply some of the rows of N_G by -1 such that each column of N_F has one 1 and one -1. Doing so, the rows of N_F sum up to 0. However, as F is a maximal forest in G and as N_G represents the cycle matroid $\mathscr{M}(G)$ of G over the reals, the columns in N_F span all the columns in N_G. Hence, the rows of N_G sum up to 0, which proves the claim. ∎

Define an orientation of G as follows. We replace each edge $uv \in E(G)$ by a directed arc. It will be directed form u to v if the entry of N_G in the row corresponding with u and the column corresponding with uv is -1. We claim that the orientation of G thus obtained has discrepancy 1. To see this let C be a circuit in G. Let $y \in GF(2)^{\{\sigma\} \cup E(G)}$ be defined by $y_e = 1$ if $e \in E(C)$; $y_e = 0$ if $e \notin E(C)$; and $y_\sigma = 1$ if and only if C is odd. Then $Sy = 0$. Hence, by Lemma 1 there exists a $\tilde{y} \in \{0, \pm 1\}^{\{\sigma\} \cup E(G)}$ such that $\tilde{y} \equiv y \pmod 2$ and $\tilde{S}\tilde{y} = 0$. But this means that the discrepancy on C is equal to

$$\sum_{e \in E(C)} \tilde{y}_e = -\tilde{y}_\sigma = 0, \pm 1,$$

which proves the Orientation Theorem.

3. Proof of the Uniqueness Theorem

We need two lemmas. If G is an undirected graph and D an orientation of G, we denote the length of a shortest path between nodes u and v in G by $d_G(u, v)$. The length of a shortest directed path from u to v in D is denoted by $d_D(u, v)$. Given a node u in G, we call D *u-conformal* if $d_G(u, v) \leqslant d_G(u, w) \leqslant d_G(u, v) + 1$ for each arc $\overrightarrow{vw} \in A(D)$.

LEMMA 3. *If D has discrepancy* 1 *and u is a node in D, then D is switching equivalent with a u-conformal orientation.*

Proof. Let G be connected and u be a node of G. Clearly, any orientation is switching equivalent with an orientation in which each node is reachable from u by a directed path. So, let us assume that D is an orientation of discrepancy 1 which already has that property. Then D contains a spanning tree T which only uses arcs $\overrightarrow{xy}$ satisfying $d_D(u, y) = d_D(u, x) + 1$. Let $\overrightarrow{uw} \in A(D)$ and let P be the path in T with endpoints v and w. On P, going from w to v, the number of forward arcs minus the number of backward arcs is $d_D(u, v) - d_D(u, w)$. Hence the circuit composed by $\overrightarrow{vw}$ and P has discrepancy $1 + d_D(u, v) - d_D(u, w)$. So for each arc $\overrightarrow{vw} \in A(D)$ we have $d_D(u, v) \leqslant d_D(u, w) \leqslant d_D(u, v) + 1$ (the second inequality follows just because $\overrightarrow{vw} \in A(D)$). From this it is not hard to see that $d_D(u, v) = d_G(u, v)$ for each node v. So the lemma follows. ∎

Let G be an undirected graph and Σ a subset of $E(G)$. We call a circuit C *ef-linking* (*with respect to* Σ) if $e, f \in E(C)$ and $E(C) \cap \Sigma \subseteq \{e, f\}$. We define an auxiliary graph G_Σ by $V(G_\Sigma) := E(G)$ and $ef \in E(G_\Sigma)$ if there exist an *ef*-linking circuit with respect to Σ.

LEMMA 4. *Let G be a 2-connected graph and $\Sigma \subseteq E(G)$. If $E(G) \backslash \Sigma$ is a spanning and connected subgraph of G, then the subgraph of G_Σ induced by the nodes in Σ is connected.*

Proof. It is easy to see that if $ef, fg \in E(G_\Sigma)$ and $f \notin \Sigma$, then $eg \in E(G_\Sigma)$. So we only need to prove that G_Σ itself is connected.

G is 2-connected, so for each pair of edges in G there exists a circuit containing both these edges. Hence, the lemma follows from:

(8) If C is a circuit in G, then all edges of C lie in the same component of G_Σ.

Assume (8) is wrong; let C be a counterexample with $|E(C) \cap \Sigma|$ as small as possible. Clearly, $|E(C) \cap \Sigma| \geqslant 2$. As $E(G) \backslash \Sigma$ is spanning and connected in G, there exist an st-path P with $E(P) \subseteq E(G) \backslash \Sigma$ and $V(C) \cap V(P) = \{s, t\}$, where s and t lie in different components of $C \backslash \Sigma$. Let C_1 and C_2 be the two circuits formed by P and an st-path in C. As $|E(C_1) \cap \Sigma|$, $|E(C_2) \cap \Sigma| < |E(C) \cap \Sigma|$ and $E(C_1) \cap E(C_2) \neq \emptyset$, all edges of C_1 and C_2 belong to the same component of G_Σ. Hence so do the edges of C—contradiction! ∎

Proof of the Uniqueness Theorem. Let G be an undirected graph with an orientation D of discrepancy 1. We may assume that G is 2-connected.

Let $u \in V(G)$. By Lemma 3, we may restrict ourselves to the case that D is u-conformal. We define $\Sigma_u := \{vw \in E(G) \mid d_G(u, w) = d_G(u, v)\}$. We call a u-conformal orientation D *nicely u-conformal* if for each pair of edges e and f in Σ_u and for each circuit C that is ef-linking with respect to Σ_u, the arcs in D corresponding with e and f are oriented in the opposite direction along C. In a u-conformal orientation we have that on each path with endpoints v and w and containing no edges in Σ_u the number of forwardly directed arcs minus the number of backwardly directed arcs is $-d_G(u, v) + d_G(u, w)$ (going from v to w). Hence:

(9) A u-conformal orientation of discrepancy 1 is nicely u-conformal.

Define the u-conformal orientation D^- obtained from D by reversing the direction of all the arcs in Σ_u. By Lemma 4, D and D^- are the only nicely u-conformal orientations. Hence, to complete the proof of the Uniqueness Theorem it suffices to show that they are switching equivalent. To see that, observe that the arcs in $A(D) \backslash \Sigma_u$ form a cut which is the disjoint union of directed cuts. Switching D on these directed cuts, followed by total switching yields D^-. ∎

4. Finding an Orientation of Discrepancy 1

The Orientation Theorem naturally rises the question for the polynomial solvability of the following problems:

Discrepancy-1: Given a directed graph D, decide whether or not it has discrepancy 1.

Orientability: Given an undirected graph G, decide whether or not it has an orientation of discrepancy 1.

Orientation: Given an undirected graph G, find an orientation of discrepancy 1 or decide that G has no such orientation.

LEMMA 5. **Discrepancy-1**, **Orientability**, *and* **Orientation** *are polynomially equivalent.*

Proof. Let $u \in V(G)$ and let D be a nicely u-conformal orientation. We call a collection $C_{e_i f_i}$ $(i = 1, ..., k)$ of $e_i f_i$-linking circuits a *linking circuit basis for* u if $e_1 f_1, ..., e_k f_k$ form a spanning tree in the subgraph of G_{Σ_u} induced by Σ_u. We call a u-conformal orientation *nice with respect to* a given linking circuit basis $C_{e_i f_i}$ $(i = 1, ..., k)$ if for each $i = 1, ..., k$, e_i and f_i have opposite orientation along C_i. In the previous section we have seen that each u-conformal orientation of discrepancy 1 is nice with respect to any linking circuit basis. Conversely, a u-conformal orientation which is nice with respect to some linking circuit basis has discrepancy 1 if and only if the underlying undirected graph has an orientation of discrepancy 1.

It is not hard to see that given u and a linking circuit basis for u, we can construct in polynomial time a u-conformal orientation which is nice with respect to the linking circuit basis. Similarly, it is easy to check in polynomial time whether a given orientation can be switched to a u-conformal orientation that is nice with respect to a given circuit basis.

From all this, we easily conclude that the three problems are polynomially equivalent. ∎

The relation between the three problems above as well as the Uniqueness Theorem corresponds—not surprisingly—with a similar situation for totally unimodular matrices (cf. Schrijver [12, p. 249]). In fact, the phenomena described here can be derived from their counterparts in the case of totally unimodular matrices.

Classes of graphs with no odd-K_4 and no odd-K_3^2 are:

— graphs that contain a node that is in each odd circuit in the graph;

— graphs that can be embedded in the plane such that all but two faces are bounded by an even circuit.

It is not hard to see that each of the graphs described above have an orientation of discrepancy 1. In [6] (cf. Gerards [7, Thm. 3.2.6]), it is proved that any graph with no odd-K_4 and no odd-K_3^2 can be decomposed into graphs of the above described two types. This yields not only a proof of the Orientation Theorem (different from the one given in Section 2) but also a polynomial-time algorithm for **Orientation** and hence also for **Discrepancy-1** and **Orientability**. This decomposition result for graphs with no odd-K_4 and no odd-K_3^2 follows from Seymour's decomposition theorem for regular matroids [13].

5. Applications of the Orientation Theorem

In this section we mention two applications of the Orientation Theorem.

5.1. *Stable Sets*

Let G be an undirected graph with no odd-K_4 as a subgraph. Then the following min–max relation holds:

(10) The maximum cardinality of a stable set in G is equal to the minimum "cost,"

$$k + \tfrac{1}{2}(|E(C_1)| - 1) + \cdots + \tfrac{1}{2}(|E(C_m)| - 1),$$

of a collection of edges $e_1, ..., e_k$, and odd circuits $C_1, ..., C_m$, such that each node in G is on one of the edges or on one of the odd circuits in this collection [4].

Note that this min–max relation extends König's min–max relation for stable sets in bipartite graphs [9, 10]. The proof of (10) as given in Gerards [4] strongly relies on the orientation theorem stated above. In case G contains no odd-K_4 but does contain an odd-K_3^2 the graph G is not 3-connected ([6]; cf. Gerards [7]). In that case an inductive argument is used. In case G has no odd-K_3^2 the orientation of discrepancy 1 makes it possible to reformulate the "covering by edges and odd circuits problem" in (10) into a circulation problem. Through this reformulation the min–max relation above easily follows.

5.2. *Homomorphism and Colouring*

Let G and H be undirected graphs. We say that G *maps into* H if there exists a map ϕ (called a *homomorphism*) from $V(G)$ to $V(H)$ such that $\phi(u)\,\phi(v) \in E(H)$ for each $uv \in E(G)$. For instance G maps into a clique of size k if and only if G is k-colorable.

Let C be an odd circuit. When does G map into C? It is easy to see that a necessary condition is that G contains no odd circuits shorter than C. Generally this condition is not sufficient. For instance, the clique on four nodes does not map into the triangle. However, in some cases the necessary condition is sufficient as well.

THEOREM [3]. *Let G be an undirected graph containing neither an odd-K_4 nor an odd-K_3^2. Then G maps into its shortest odd circuit.*

Proof. (Alexander Schrijver. An elementary but more complicated proof, independent of the Orientation Theorem, is given in Gerards [3].) Let $D=(V(G), A(D))$ be an orientation of G of discrepancy 1. For each arc in $A(D)$ going from u to v we add a new arc going from v to u. The collection of these reverse arcs is denoted by $R(D)$. So if we denote an arc from u to v by $\overrightarrow{uv}$, then $R(D):=\{\overrightarrow{uv} \mid \overrightarrow{vu} \in A(D)\}$. We define the directed graph D^+ as follows: $V(D^+):=V(D)=V(G)$, and $A(D^+):=A(D)\cup R(D)$. On $A(D^+)$ we define a length function w by

$$w():=\begin{cases} k+1 & \text{if} \quad a\in A(D) \\ -k & \text{if} \quad a\in R(D). \end{cases}$$

As $A(D)$ has discrepancy 1, and as G has no odd circuits containing less than $2k+1$ edges, no directed circuit in D^+ has negative length with respect to w. Hence there exists an integer valued function π on $V(D^+)=V(G)$ satisfying

$$\pi(v)-\pi(u)\leqslant w(\overrightarrow{uv}) \qquad \text{for each} \quad \overrightarrow{uv}\in A(D^+).$$

Indeed, fix $u_0\in V(G)$. With w as length function let, for each $u\in V(G)$, $\pi(u)$ be the length of the shortest directed path from u_0 to u. Then π satisfies the inequalities above. Hence we have

$$k\leqslant |\pi(v)-\pi(u)| \leqslant k+1 \qquad \text{for each} \quad uv\in E(G).$$

So

$$|2\pi(v)-2\pi(u)| \equiv \pm 1 \qquad (\text{mod } 2k+1) \text{ for each } uv\in E(G).$$

Assume the nodes of the circuit C are labeled $v_1, ..., v_{2k+1}$ (in cyclic order). For each $u\in V(G)$, let $l(u)\in\{1, ..., 2k+1\}$ be such that $l(u)\equiv 2\pi(u)$ (mod $2k+1$). Then the function f defined on $V(G)$ by $f(u):=v_{l(u)}$ for each $u\in V(G)$, maps G into C. ∎

An easy corollary of this result is that graphs with no odd-K_4 and no odd-K_3^2 are 3-colorable (as each odd circuit maps into the triangle). In fact, as has been proved by Catlin [2], graphs with no odd-K_4 are 3-colorable. Catlin's result follows now easily because graphs with no odd-K_4 but with an odd-K_3^2 have a 2-node cutset ([6]; cf. Gerards [7]), so admit an inductive argument.

6. A Dual Version of the Theorem

András Frank posed the question whether there exists a dual version of the orientation theorem, where "circuits" are replaced by "cuts." Indeed such a result exists. We state it here without proof (cf. Gerards [7]).

Let G be an undirected graph. We call a subset U of $V(G)$ *elementary* if both U and $V(G)\backslash U$ induce connected subgraphs of G. U is called *odd* if there are an odd number of edges leaving U.

THEOREM. *Let G be a connected graph. Then* (i) *and* (ii) *below are equivalent*:

(i) *G has an orientation D such that each elementary subset U of $V(G)$ satisfies*

$$\left| |\{\overrightarrow{uv} \in A(D) \mid u \in U, v \notin U\}| - |\{\overrightarrow{uv} \in A(D) \mid u \notin U, v \in U\}| \right| \leqslant 1;$$

(ii) *neither one of the following holds*:

(a) *$V(G)$ can be partitioned into odd elementary subsets U_1, U_2, U_3, and U_4, such that every pair among U_1, U_2, U_3, and U_4, is connected by an edge in G.*

(b) *$V(G)$ can be partitioned into elementary subsets U_1, U_2, U_3, V_1, and V_2, such that U_1, U_2, and U_3 are odd and every pair U_i, V_j is connected by an edge in G.*

Remark. Note that if G is planar, then the equivalence of (i) and (ii) above easily follows from the Orientation Theorem by planar duality.

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